Simplified derivation of the Bethe-ansatz equations for the Dicke model

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We present an elementary derivation of the exact solution (Bethe-ansatz equations) of the Dicke model, using only commutation relations and an informed ansatz for the structure of its eigenstates.

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In 1954, Dicke showed that a model describing a set of two-level systems coupled to a quantized electromagnetic mode leads to a superradient effect.[1] Generalizations to multicomponent systems naturally appear in various experimentally relevant contexts.[2,3] Other generalizations involve a spatially extended photonic field,[4] or itinerant two-level systems,[5] motivated by an experiment on cold atoms in a two-dimensional lattice coupled to an optical resonator.[6]

The Bethe-ansatz solution for the Dicke model with inhomogeneous excitation energies was originally obtained by Gaudin[7] as a side result of solving the central spin problem. Using a variational method that results in complex algebraic computations, he showed that the solution of the central spin problem is equivalent to the Bethe-ansatz solution of the BCS problem derived by Richardson.[8] By expanding the Bethe-ansatz equations for the central spin model in the limit of large central spin, Gaudin obtained corresponding equations for the Dicke model.[7] Though this procedure solves the original problem, the derivation is computationally complex, and thus not easily extended to other, related models.

The purpose of this Brief Report is to provide an elementary derivation that starts from the original Dicke model, in the hope that our simplified treatment might pave the way toward finding similar solutions to generalized Dicke models. We follow a method suggested by Richardson[9] for the BCS model and presented in Refs. 10 and 11. This method exploits the observation that the structure of the exact eigenstates of the Dicke model is similar to that of an auxiliary model, involving only bosons. The only difference is that the eigenvalue equations that determine the quasienergies characterizing these states become more complicated for the Dicke model: they turn into Gaudin's Bethe-ansatz equations, which we derive here using only commutation relations.

The inhomogeneous Dicke model describes a set of nonidentical two-level systems with excitation energies ϵ_j and a single photon mode with frequency ω, coupled with interaction strength g

$$H=\omega b^{\dagger}b+\sum_{j=1}^{N}\epsilon_j\left(S_j^z+\frac{1}{2}\right)+g\sum_{j=1}^{N}(S_j^{+}b+S_j^{-}b^{\dagger}). \qquad (1)$$

The spin-$\frac{1}{2}$ operators satisfy $(S_j^{\pm})^2=0$ and

$$[S_i^-,S_j^+]=-2S_j^z\delta_{ij}, \quad [S_i^z,S_j^{\pm}]=\pm S_j^{\pm}\delta_{ij} \qquad (2)$$

while the boson operators satisfy $[b,b^{\dagger}]=1$.

Let $|\mathrm{Vac}\rangle$ be the "vacuum" state containing no boson excitations and all spins down, i.e., $b|\mathrm{Vac}\rangle=S_j^-|\mathrm{Vac}\rangle=0$. H commutes with the operator $b^{\dagger}b+\Sigma_{j=1}^{N}S_j^z$, which counts the number of excitations relative to $|\mathrm{Vac}\rangle$. Thus, H eigenstates can be constructed by acting on $|\mathrm{Vac}\rangle$ with (products of) linear combinations of S_j^+ and $b^{\dagger}$ operators, of the general (unnormalized) form

$$B_{\nu}^{\dagger}=b^{\dagger}+\sum_{j=1}^{N}A_{\nu j}S_j^{+}, \qquad (3)$$

where the coefficients $A_{\nu j}$ are to be determined. For an eigenstate with n excitations relative to $|\mathrm{Vac}\rangle$ we thus make the ansatz (following Refs. 10 and 11)

$$|\Psi_n\rangle=P_1^n|\mathrm{Vac}\rangle, \qquad (4)$$

where we use the shorthand notation (for $n'\le n$)

$$P_{n'}^{n}=\prod_{\nu=n'}^{n}B_{\nu}^{\dagger} \qquad (5)$$

for a product of $B^{\dagger}$'s (for $n'>n$, we set $P_{n'}^{n}=1$). For later use, note that such products satisfy the composition rule $P_{n'}^{\nu}P_{\nu+1}^{n}=P_{n'}^{n}$ for $n'\le\nu<n$.

We require that $H|\Psi_n\rangle=\mathcal{E}_n|\Psi_n\rangle$. Commuting H past P_1^n to the right and using $H|\mathrm{Vac}\rangle=0$, we obtain

$$(\mathcal{E}_nP_1^n-[H,P_1^n])|\mathrm{Vac}\rangle=0. \qquad (6)$$

Using the general operator identity

$$[X,P_{n'}^{n}]=\sum_{\nu=n'}^{n}P_{n'}^{\nu-1}[X,B_{\nu}^{\dagger}]P_{\nu+1}^{n}. \qquad (7)$$

Equation (6) can be written as

$$\left(\mathcal{E}_nP_1^n-\sum_{\nu=1}^{n}P_1^{\nu-1}[H,B_{\nu}^{\dagger}]P_{\nu+1}^{n}\right)|\mathrm{Vac}\rangle=0. \qquad (8)$$

The requisite commutator is given by

$$[H,B_{\nu}^{\dagger}]=\sum_{j=1}^{N}(A_{\nu j}\epsilon_j+g)S_j^{+}+(\omega-2gX_{\nu})b^{\dagger}, \qquad (9)$$

where $X_{\nu}=\Sigma_{j=1}^{N}A_{\nu j}S_j^z$. By making the choice

$$A_{\nu j}=\frac{g}{E_{\nu}-\epsilon_j}, \qquad (10)$$

where the parameters E_{ν} will be called quasienergies, Eq. (9) can be brought into the simplified form

$$[H,B_\nu^\dagger] = E_\nu B_\nu^\dagger + (\omega - E_\nu - 2gX_\nu)b^\dagger. \tag{11}$$

Inserting this into Eq. (8) and identifying the eigenenergy with the sum on quasienergies, $\mathcal{E}_n = \sum_{\nu=1}^n E_\nu$, yields

$$\sum_{\nu=1}^{n} P_1^{\nu-1}(\omega - E_\nu - 2gX_\nu)P_{\nu+1}^n b^\dagger|\mathrm{Vac}\rangle = 0. \tag{12}$$

To make sense of this condition consider, for a moment, an auxiliary, purely bosonic model, obtained from the Dicke Hamiltonian (1) by replacing S_j^+, S_j^-, and $(S_j^z + \frac{1}{2})$ by $b_j^\dagger$, b_j, and $b_j^\dagger b_j$, respectively, with $[b_i, b_j^\dagger] = \delta_{ij}$. Repeating the above analysis yields only one change: since $[b_j, b_j^\dagger]$ gives 1 instead of $[S_j^-, S_j^+]$ giving $-2S_j^z$, the *operator* X_ν in Eq. (9) is replaced by the *c number* $x_\nu = -\frac{1}{2}\sum_{j=1}^N A_{\nu j}$. Thus Eq. (12) can be satisfied by requiring that $\omega - E_\nu - 2gx_\nu = 0$ for all ν. Via Eq. (10) this implies $\omega - E_\nu + \sum_{j=1}^N g^2/(E_\nu - \epsilon_j) = 0$, which determines the E_ν. This equation can also be obtained by making the ansatz $H = \sum_\nu E_\nu B_\nu^\dagger B_\nu$ and demanding that $[H, B_\nu^\dagger] = E_\nu B_\nu^\dagger$. For this auxiliary model the $B_\nu^\dagger$ thus describe independent single-particle excitations and the quasienergies E_ν are their eigenenergies.

Let us now return to the Dicke model, where X_ν is an operator, so that we have to work a little (but not much!) harder to satisfy Eq. (12). To this end, commute X_ν past $P_{\nu+1}^n$ to the right and use $X_\nu|\mathrm{Vac}\rangle = x_\nu|\mathrm{Vac}\rangle$, to obtain

$$\sum_{\nu=1}^{n} P_1^{\nu-1}P_{\nu+1}^n(\omega - E_\nu - 2gx_\nu)b^\dagger|\mathrm{Vac}\rangle \tag{13a}$$

$$= 2g\sum_{\nu=1}^{n} P_1^{\nu-1}[X_\nu, P_{\nu+1}^n]b^\dagger|\mathrm{Vac}\rangle. \tag{13b}$$

To simplify the second line, use Eq. (7) and the relation

$$[X_\nu, B_\mu^\dagger] = -g\frac{B_\nu^\dagger - B_\mu^\dagger}{E_\nu - E_\mu}, \tag{14}$$

which follows from $A_{\nu j}A_{\mu j} = -g(A_{\nu j} - A_{\mu j})/(E_\nu - E_\mu)$ to write $\sum_{\nu=1}^n P_1^{\nu-1}[X_\nu, P_{\nu+1}^n]$ as

$$-g\sum_{\nu=1}^{n} P_1^{\nu-1}\sum_{\mu=\nu+1}^{n} P_{\nu+1}^{\mu-1}\frac{B_\nu^\dagger - B_\mu^\dagger}{E_\nu - E_\mu}P_{\mu+1}^n \tag{15a}$$

$$= \sum_{\nu=1}^{n} P_1^{\nu-1}P_{\nu+1}^n\sum_{\mu=1,\mu\neq\nu}^{n}\frac{g}{E_\nu - E_\mu}. \tag{15b}$$

Equation (15b) follows by relabeling $\nu \leftrightarrow \mu$ in the $B_\nu^\dagger$ term of Eq. (15a). Inserting Eq. (15b) into Eq. (13b), we note that Eq. (13) is satisfied provided that the n quasienergies E_ν obey the following n coupled equations:

$$\omega - E_\nu + \sum_{j=1}^{N}\frac{g^2}{E_\nu - \epsilon_j} = \sum_{\mu=1,\mu\neq\nu}^{n}\frac{2g^2}{E_\nu - E_\mu}. \tag{16}$$

These are the celebrated Bethe-ansatz equations for the Dicke model, first obtained by Gaudin.[7] The fact that the right-hand side couples the equations for different E_ν together presents the additional complication arising for the Dicke model in comparison to the above-mentioned auxiliary boson model. It implies that the $B_\nu^\dagger$ do *not* describe independent single-particle excitations, since the value of any E_ν depends on that of all others.

Generally Eq. (16) has to be solved numerically. For sufficiently small n, however, the original model Eq. (1) can be diagonalized directly by solving the eigenvalue problem in the basis of uncoupled bosonic and spin eigenstates[12] instead of the basis Eq. (4).

It is straightforward to expand the normalization factors of Gaudin eigenstates[7] and verify that $|\langle\Psi_n|\Psi_n\rangle|^2 = \det\hat{M}$, where $\hat{M}$ is an $n\times n$ matrix with elements $M_{\nu\nu} = 1 + \sum_{j=1}^N A_{\nu j}^2 - 2\sum_{\mu=1,\mu\neq\nu}^n A_{\mu\nu}^2$ and $M_{\mu\nu} = 2A_{\mu\nu}^2$, and we used the shorthand $A_{\nu\mu} = g/(E_\nu - E_\mu)$.

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